Solution to First Proof Second Batch Problem C3

Martin Hairer, EPFL and Imperial

July 1, 2026

Question

Fix a compact Lie group $\mathbf{G}$ with Lie algebra $\mathfrak{g}$, endowed with an Ad-invariant norm. We define invariant $\mathcal{C}^{-\kappa}_{\mathrm{inv}}$ "norms" on smooth $\mathbf{G}$-equivariant connections B_0 on the trivial $\mathbf{G}$-bundle over the torus $\mathbb{T}^2$ by

$$\|B_0\|_{\mathrm{inv},-\kappa} = \sup_{s\in(0,1]} s^{\frac{\kappa}{2}} \big(s^{\frac{1}{2}} \|F(s)\|_{L^\infty} + s\|\mathrm{D}F(s)\|_{L^\infty}\big) \;. \tag{1}$$

Here $F(s) = F_{B(s)}$ denotes the curvature tensor for the solution $B(s)$ at time s to the Yang–Mills gradient flow with initial condition B_0 and $\mathrm{D}F(s)$ denotes the covariant derivative of $F(s)$ with respect to the connection $B(s)$. The quantity given by (1) is of course not a genuine norm since the relation $B_0 \mapsto F$ is nonlinear. Recall also that the group of smooth functions $g\colon \mathbb{T}^2 \to \mathbf{G}$ acts on such connections B_0 by

$$g \bullet B_0(x) = \mathrm{Ad}_{g(x)} B_0(x) - dg(x)\, g(x)^{-1} \;.$$

Give a proof of the following theorem.

Theorem 1. *Given $\kappa \in (0, \frac{1}{2})$, there exist constants $c, C > 0$ such that the following is true. For every smooth connection B_0 as above, there exists a gauge transformation g such that*

$$\|g \bullet B_0\|_{\mathcal{C}^{-\kappa}} + \sup_{s\in(0,1]} s^\kappa \|\mathrm{div}(g \bullet B(s))\|_{L^\infty} \le C(1 + \|B_0\|_{\mathrm{inv},-\kappa})^c \;. \tag{2}$$

Answer

The main workhorse for the proof of Theorem 1 is the following quantitative form of Uhlenbeck compactness in the case when $\mathbf{G}$ is simply connected. We will later show how to reduce ourselves to that case.

Proposition 2. In the setting of Theorem 1 but for simply connected $\mathbf{G}$, there exist constants $c, C > 0$ such that, for every smooth connection B with curvature F, there exists a gauge transformation g such that $\|g \bullet B\|_{L^\infty} \le C(1 + \|F\|_{L^\infty})^c$.

Remark 3. At first glance, one has the impression that, by choosing p large and using the embeddings $L^\infty \subset L^p$ (for F) and $W^{1,p} \subset L^\infty$ (for $g \bullet B$), this should be an immediate consequence of keeping track of constants in the proof of Uhlenbeck's weak compactness theorem [**?**, **?**], see the detailed exposition in [**?**, Thm 7.1]. This does unfortunately not seem to be the case for the following reason. When considering a sequence of connections, that proof first exhibits a candidate limit which is such that, for a covering of the torus by small patches, one can find on each patch a gauge transformation leading to a good control of the supremum norm of that limit. The argument then proceeds to showing that, for connections with curvature close to that of the limit, one can find gauge transformations such that their *differences* with the limit is controlled in $W^{1,p}$. In particular, one never needs to get any quantitative control on the limit in any global gauge! We will therefore use a different patching argument, which uses in a crucial way the fact that our base space is a two-dimensional manifold. (Our argument would still work in three dimensions, but it breaks down in dimension four.)

Before turning to the proof of this result, we state two relatively straightforward extension results, the second of which makes crucial use of the fact that $\mathbf{G}$ is simply connected.

Lemma 4. For every Lipschitz continuous $f_0\colon [0,1] \to \mathbf{G}$ there exists $f\colon [0,1]^2 \to \mathbf{G}$ such that $f(0,x) = f_0(x)$ and $f(1,x) = e$ for all $x \in [0,1]$. Furthermore, there exists a constant C such that $\mathrm{Lip}(f) \le C(1+\mathrm{Lip}(f_0))$.

Proof. Since $\mathbf{G}$ is compact, the torus theorem [**?**, Thm 11.9] shows that there exists $h \in \mathfrak{g}$ such that $\exp(h) = f_0(0)$. It then suffices to set for example $f(t,x) = f_0((1-t)x)\exp(-th)$ and to note that since $\mathbf{G}$ is compact there exists a constant C such that h can always be chosen such that $|h| \le C$. $\square$

Lemma 5. Assume that $\mathbf{G}$ is compact and simply connected. Then, for every Lipschitz continuous $f_0\colon S^1 \to \mathbf{G}$, there exists $f\colon [0,1]\times S^1 \to \mathbf{G}$ such that $f(0,x) = f_0(x)$ and $f(1,x) = e$ for all $x \in S^1$. Furthermore, there exists a constant C such that $\mathrm{Lip}(f) \le C(1+\mathrm{Lip}(f_0))$.

Proof. While the existence of such an f is the definition of simple connectedness, we provide an explicit construction which yields the claimed quantitative bound. Let $K>0$ be such that $\exp\colon \mathfrak{g} \to \mathbf{G}$ is a diffeomorphism on the ball $B(K)$ of radius K and such that both $\exp$ and $\exp^{-1}$ have Lipschitz constant less than 2 on $B(K)$, respectively $\exp(B(K))$. (In particular $\exp^{-1}$ is a smooth function on the ball of radius $K/2$ around the neutral element of $\mathbf{G}$.) Fix a finite set $V \subset \mathbf{G}$ that is sufficiently dense such that every $g \in \mathbf{G}$ is at distance at most $K/8$ from some element $v(g) \in V$. For every $v \in V$, we also fix $h_v \in \mathfrak{g}$ with $|h_v| \le C$ such that $v = \exp(h_v)$. Finally, for any pair $(u,v) \in V$ with $d(u,v) \le K/2$, we write $g_{uv}\colon [0,1] \to \mathbf{G}$ for the Lipschitz continuous loop given by

$$g_{uv}(x) = \begin{cases} \exp(3xh_u) & \text{if } 3x \in [0,1], \\ \exp\big((3x-1)\exp^{-1}(vu^{-1})\big)u & \text{if } 3x \in [1,2], \\ \exp((3-3x)h_v) & \text{if } 3x \in [2,3]. \end{cases} \tag{3}$$

Since $\mathbf{G}$ is simply connected, we can extend g_{uv} to a Lipschitz continuous map $g_{uv}\colon [0,1]^2 \to \mathbf{G}$ such that $g_{uv}(0,x)$ is given by (3) and such that $g_{uv}(t,0) = g_{uv}(t,1) = g_{uv}(1,x) = e$. (The fact that a continuous such map exists is the definition of simple connectedness and it can be made continuous by a 'Lipschitz' version of Whitney's approximation theorem, see for example [**?**, Chap. 6].)

Writing $M = \mathrm{Lip}(f_0)$, partition now S^1 into $O(M)$ intervals $I_k = [x_k, x_{k+1}]$ of length at most $K/(4M)$, so that in particular, for every k, $d(f_0(x), f_0(y)) \le K/4$ for all $x,y \in I_k$. Let furthermore $v_k \in V$ be such that $d(v_k, f_0(x_k)) \le K/8$, so that $d(v_k, v_{k+1}) \le K/2$, and set $h_k = h_{v_k}$. We then write $f^{(1)}\colon S^1 \to \mathbf{G}$ for the (unique) continuous map such that $f^{(1)}(x_k) = v_k$ and such that its restriction to each I_k is a constant speed geodesic of length less than $K/2$. In particular, for $y \in [x_k, (x_k+x_{k+1})/2]$ one has

$$d(f^{(1)}(y), f_0(y)) \le d(v_k, v_{k+1}) + d(v_k, f_0(x_k)) + d(f_0(x_k), f_0(y)) \le \frac{K}{4} + \frac{K}{8} + \frac{K}{8} \le \frac{K}{2}\;, \tag{4}$$

which holds for all y by possibly swapping the roles of v_k and v_{k+1}. Note that one has again $\mathrm{Lip}(f^{(1)}) = O(M)$. For $t \in [0,1/3]$, we then set

$$f(t,y) = \exp\big(3t\exp^{-1}(f^{(1)}(y)f_0(y)^{-1})\big)f_0(y)\;.$$

Note that one has indeed $\mathrm{Lip}(f) = O(M)$ as a consequence of the fact that, for $\mathbf{G}$-valued functions, one has $\mathrm{Lip}(fg) \lesssim \mathrm{Lip}(f) + \mathrm{Lip}(g)$, as well as the fact that both $\exp$ and $\exp^{-1}$ have Lipschitz constant 2 on their domain of application by (4) and the definition of K.

For $t \in [1/3, 2/3]$ and $x \in I_k$, we then set $f(t,x) = f_k^{(2)}(3t-1, (y-x_k)/|I_k|)$, where $f_k^{(2)}$ is defined by

$$f_k^{(2)}(t,y) = \begin{cases} \exp((3x+1-t)h_k) & \text{if } 3x \in [0,t], \\ \exp\big(\frac{3x-t}{3-t}\exp^{-1}(v_{k+1}v_k^{-1})\big)v_k & \text{if } 3x \in [t, 3-t], \\ \exp((4-3x-t)h_{k+1}) & \text{if } 3x \in [3-t, 3]. \end{cases}$$

Since $f_k^{(2)}(0,\cdot)$ is the constant speed geodesic from v_k to v_{k+1} and since the $f_k^{(2)}$ are all Lipschitz continuous with Lipschitz constant of order 1 (independent of M), this definition of f is still Lipschitz continuous with $\mathrm{Lip}(f) = O(M)$.

Furthermore we note that, for every k, one has $f_k^{(2)}(1,\cdot) = g_{v_k,v_{k+1}}$ defined as in (3). For $t \in [2/3, 1]$ and $x \in I_k$, it therefore suffices to set $f(t,x) = g_{v_k,v_{k+1}}(3t-2, (y-x_k)/|I_k|)$, which concludes the proof. □

Proof of Proposition 2. We set $M = 1 + \|F\|_{L^\infty}$. By the proof of weak Uhlenbeck compactness as exposited in [?, Chap. 7], there exists an exponent q such that, if one divides the torus into a collection A of $O(M^{2q})$ overlapping squares of sidelength about M^{-q}, one can find for each $a \in A$ a gauge transformation $g_a\colon a \to \mathbf{G}$ such that $\|g_a \cdot B \restriction a\|_{W^{1,p}} \lesssim 1$. In particular one has $\|g_a \cdot B \restriction a\|_{L^\infty} \lesssim 1$ by choosing $p > 2$. To be more precise regarding the locations of the squares, we place the centers of the squares on a regular grid of grid size $\delta \approx M^{-2q} \in \mathbb{N}^{-1}$ and we choose them of sidelength $4\delta/3$, so that neighbouring squares overlap on a strip of size $\frac{\delta}{3} \times \delta$.

For any overlapping pair of squares $a, b \in A$, set $g_{ab} \stackrel{\text{def}}{=} g_a g_b^{-1}$. By [?, Lemma A.8], the $\mathfrak{g}$-valued one forms $h_{ab} = dg_{ab}g_{ab}^{-1}$ satisfy $\|h_{ab}\|_{W^{1,p}} \leq 1$. In particular, it follows that the transition functions g_{ab} are Lipschitz continuous with Lipschitz constants of order 1, independently of M.

Assume now that one has $a = I_a \times J$ and $b = I_b \times J$ with the overlap consisting of the rectangle $I \times J$ with $I = I_a \cap I_b$ (to fix ideas, say that a is to the left of b). By rescaling space suitably, we can assume that $I = J = [0,1]$ and that g_{ab} is defined on $[0,1]^2$. We then use Lemma 4 to construct a Lipschitz function $f\colon [0,1]^2 \to \mathbf{G}$ which agrees with g_{ab} on $\{0\} \times [0,1]$ and equals the neutral element e on $\{1\} \times [0,1]$ and we set $g_{a\cap b}(x) = f(x)g_b(x)$. If we now set $\hat{a} = a \cup b$ and define $g_{\hat{a}}$ by g_a on $a \setminus b$, g_b on $b \setminus a$ and $g_{a\cap b}$ on $a \cap b$, we see that

$$\|g_{\hat{a}} \cdot B \restriction \hat{a}\|_{L^\infty} \leq \|g_a \cdot B \restriction a\|_{L^\infty} \vee \big(\|g_b \cdot B \restriction b\|_{L^\infty} + \mathrm{Lip}(f)\big)\;.$$

Since g_{ab} has Lipschitz constant of order 1, it furthermore follows from the construction of Lemma 4 that $\mathrm{Lip}(f) \lesssim M^q$.

We now divide the collection A into horizontal strips and, for each of the strips, we sequentially perform the above construction for all pairs of neighbouring squares within the strip. At this stage, we then have a collection $\hat{A}$ of strips and, for each strip $a \in \hat{A}$, a gauge g_a such that $\|g_a \cdot B \restriction a\|_{L^\infty} \lesssim M^q$. Furthermore, we claim that the transition functions g_{ab} between neighboring strips are such that $\mathrm{Lip}(g_{ab}) \lesssim M^q$. This is because they either consist of the original transition functions (which have Lipschitz constants of order 1) or they are of the form $f_1 g_{a_1} g_{a_2}^{-1} f_2^{-1}$ for f_i some of the interpolating functions constructed above. Since one has $\mathrm{Lip}(fg) \lesssim \mathrm{Lip}(f) + \mathrm{Lip}(g)$ and since the interpolating functions have Lipschitz constants of order at most M^q, the claim follows.

It now remains to patch the strips together. This is done again by the same construction, but this time we have to use Lemma 5 since the boundaries of the strips consist of circles. We furthermore lose another factor M^q since this is the width of the overlaps of neighbouring strips. Once all the strips have been patched together, this indeed yields a gauge g such that $\|g \cdot B\|_{L^\infty} \lesssim M^{2q}$, as claimed. □

Before we turn to the proof of Theorem 1, we state two additional results which are crucial for removing the assumption that $\mathbf{G}$ is simply connected.

Lemma 6. The conclusion of Proposition 2 also holds in the special case $\mathbf{G} = U(1)$.

Proof. Setting $\mathbf{G} = \mathbb{R}/\mathbb{Z}$ and $\mathfrak{g} = \mathbb{R}$, one identifies as usual gauge transformations with functions $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that $g(x+y) - g(x) \in \mathbb{Z}$ for $y \in \mathbb{Z}^2$ and we have $g \bullet B = B + \nabla g$. Writing $k \in \mathbb{Z}^2$ for an element such that $|k - \bar{B}| \le 1$ where $\bar{B}$ is the spatial average of B, it then suffices to choose

$$g(x) = -\Delta^{-1} \operatorname{div} B(x) - kx \;.$$

The claim then follows from the fact that, by the Helmholtz decomposition and the identity $F_{12} = \operatorname{curl} B$, one has $g \bullet B = (\bar{B} - k) + \nabla^\perp \Delta^{-1} F_{12}$. □

Given a Lie group of the type $\mathbf{G} = \mathbf{G}_1 \times \mathbf{G}_2$, we write its elements as $g = g_1 \oplus g_2$ and similarly for its Lie algebra. We then have the following.

Lemma 7. Given a connection one-form $B = B_1 \oplus B_2$ and a gauge transformation $g = g_1 \oplus g_2$, one has

$$g \bullet B = (g_1 \bullet B_1) \oplus (g_2 \bullet B_2) \;.$$

Proof. It follows immediately from the fact that $\mathrm{Ad}_g h = (\mathrm{Ad}_{g_1} h_1) \oplus (\mathrm{Ad}_{g_2} h_2)$ and $dg\, g^{-1} = dg_1\, g_1^{-1} \oplus dg_2\, g_2^{-1}$. □

We now have all the ingredients in place for the proof of the main result.

Proof of Theorem 1. In a first step, we show that Proposition 2 holds for arbitrary compact Lie groups $\mathbf{G}$, not just simply connected ones. We may however assume without loss of generality that $\mathbf{G}$ is connected since we can always decide to only consider gauge transformations taking values in the connected component of the identity element. It then follows from the structure theorem for compact Lie groups [**?**, Thm V.8.1] that $\mathbf{G} \simeq (T \times K)/Z$, where T is a torus, K is a simply-connected, semisimple, compact Lie group, and Z is a finite subgroup of the center of $T \times K$. We now let $\mathbf{H} \coloneqq T \times K$ and let $\mathfrak{h}$ be its Lie algebra. Since Z is finite, $\mathbf{G}$ admits a fundamental domain in $\mathbf{H}$ with non-empty interior, so that one has a canonical identification $\mathfrak{h} \simeq \mathfrak{g}$. In particular, writing $\pi\colon \mathbf{H} \to \mathbf{G}$ for the canonical projection and setting $g = \pi \circ h$ for some smooth $h\colon \mathbb{T}^2 \to \mathbf{H}$, one has

$$\mathrm{Ad}_g = \mathrm{Ad}_h \;, \qquad dg\, g^{-1} = dh\, h^{-1} \;. \tag{5}$$

By Proposition 2, Lemma 6, and Lemma 7, given an $\mathfrak{h}$-valued connection one-form B, we can find an $\mathbf{H}$-valued gauge transformation h such that the bound $\|h \bullet B\|_{L^\infty} \le C(1 + \|F\|_{L^\infty})^c$ holds. Interpreting B as a $\mathfrak{g}$-valued connection, it then follows at once from (5) that the conclusion of Proposition 2 holds with $g = \pi \circ h$.

In a second step, we show that, writing $M = 1 + \|B_0, \Phi_0\|_{\mathrm{inv},-\kappa}$, there exists a constant c (independent of M) and a gauge transformation g such that $\|g \bullet B_1\|_{\mathcal{C}^1} \lesssim M^c$.

For this, note first that by Proposition 2, we can find gauge transformations g_τ such that, uniformly over $\tau \in [1/2, 1]$, one has $\|g_\tau \bullet B_\tau\|_{L^\infty} \lesssim M^c$. Setting $\hat{B}_0 = g_\tau \bullet B_\tau$ for some fixed τ, consider then the Yang–Mills–DeTurck heat flow $\hat{B}$ with initial condition $\hat{B}_0$. Simple scaling arguments show that this equation is locally well-posed for initial conditions $\hat{B}_0 \in \mathcal{C}^\alpha$ for every $\alpha > -1/2$ which in particular includes $\hat{B}_0 \in L^\infty$. Since the nonlinearity for this equation is a linear combination of terms of type $B\partial B$ and B^3, a standard Picard iteration argument for semilinear parabolic PDEs shows that, after a time t of order M^{-4c}, one has $\|\hat{B}_t\|_{\mathcal{C}^k} \lesssim M^{c(1+2k)}$. By the gauge equivalence of the Yang–Mills–DeTurck and the Yang–Mills heat flow (see for example [**?**] or [**?**, Sec. 4.1]), there exist gauge transformations $\hat{g}_t$ such that $\hat{B}_t = \hat{g}_t \bullet (g_\tau \bullet B_{\tau+t})$. It now suffices to choose $\tau = 1 - CM^{-4c}$ for a suitable constant so that one can set $\tau + t = 1$, thus yielding the desired bound $\|g \bullet B_1\|_{\mathcal{C}^1} \lesssim M^{3c}$ for $g = \hat{g}_t \cdot g_\tau$.

From now on, we will allow ourselves to change the precise value of the exponent c from one step to the next in our proofs. We furthermore place ourselves in the (time independent!) gauge g we just constructed, so we write B_0 and $B(s)$ rather than $g \bullet B_0$ and $g \bullet B(s)$ from now on, and similarly for Φ_0 and $\Phi(s)$. We also now take for granted the bounds we just obtained, namely

$$\sup\{\|B(1)\|_{L^\infty},\, \|\partial B(1)\|_{L^\infty}\} \lesssim M^c\;. \tag{6}$$

(We will sometimes write 'time' as an argument of B and sometimes as a subscript. We hope that this will not cause confusion: time values will always be s, t, while spatial indices will be j, ℓ, etc.)

The idea then is to exploit the bounds at $t = 1$ and on F to integrate the PDE $\partial_t B_\ell = \mathrm{D}_k F_{k\ell}$ *backwards* in time from time 1 in order to obtain (possibly singular) bounds on B in various non-covariant regularity norms. Setting

$$B_\ell(t) = B_\ell(1) - \int_t^1 \mathrm{D}^k F_{k\ell}(s)\,ds\;,$$

and using the bound $\|\mathrm{D}^k F_{k\ell}(s)\| \lesssim M s^{-1-\frac{\kappa}{2}}$, we immediately obtain from (6)

$$\|B(t)\|_{L^\infty} \lesssim M^c t^{-\kappa/2}\;. \tag{7}$$

Furthermore, if we test B against a localised ($\mathfrak{g}$-valued) test function ϕ_x^λ, we can write

$$\langle \phi_x^\lambda, B_0\rangle = \langle \phi_x^\lambda, B(\lambda^2)\rangle + \int_0^{\lambda^2} \langle \mathrm{D}^k \phi_x^\lambda, F_{k\ell}(s)\rangle\,ds\;.$$

As a consequence of (7) and since the covariant derivative appearing in the last expression is taken with respect to the connection $B(s)$, we immediately obtain

$$\|\mathrm{D}^k \phi_x^\lambda\|_{L^1} \lesssim \lambda^{-1} + M^c s^{-\kappa/2}\;,$$

so that

$$|\langle \phi_x^\lambda, B_0\rangle| \lesssim M^c \lambda^{-\kappa} + \int_0^{\lambda^2} M s^{-\frac{1+\kappa}{2}} (\lambda^{-1} + M^c s^{-\kappa/2})\,ds \lesssim M^c \lambda^{-\kappa}\;,$$

which yields the desired bound $\|B_0\|_{\mathcal{C}^{-\kappa}} \lesssim M^c$.

To bound $\operatorname{div} B$ we use the fact that $[\partial_t, \mathrm{D}_j] = \mathrm{ad}_{\partial_t B_j} = \mathrm{ad}_{\mathrm{D}_k F_{kj}}$, so that

$$\partial_t \mathrm{D}_j B_\ell = -\mathrm{D}_j \mathrm{D}^k F_{k\ell} + [\mathrm{D}^k F_{kj}, B_\ell]\;,$$

In the special case $j = \ell$, since F is antisymmetric and $[\mathrm{D}_\ell, \mathrm{D}_k] = \mathrm{ad}_{F_{k\ell}}$, one has

$$\partial_t \mathrm{D}^\ell B_\ell = [\mathrm{D}^k F_k{}^\ell, B_\ell]\;. \tag{8}$$

Using again (7), the right-hand side of this expression is bounded by $M^c t^{-1-\kappa}$, yielding the bound

$$\|\operatorname{div} B\|_{L^\infty} = \|\mathrm{D}^\ell B_\ell\|_{L^\infty} \lesssim M^c t^{-\kappa}\;, \tag{9}$$

as desired.

Finally, in order to bound the $\mathcal{C}^{-\kappa}$ norm of Φ_0, we use the bound

$$\|\Phi_0\|_{\mathcal{C}^{-\kappa}} \lesssim \sup_{t \in (0,1]} t^{\kappa/2} \|P_t \Phi_0\|_{L^\infty}\;.$$

The proof of this fact is a simple variant on the argument given for example in [**?**, Thm 12.4]. In order to bound $P_t\Phi_0$ we use the expression of Δ_B in coordinates (see for example [**?**, Eq. 1.7]) and the variation of constants formula to write

$$P_t \Phi_0 = \Phi(t) - \int_0^t P_{t-s}\big((\operatorname{div} B)\Phi - B_j^2 \Phi - 2\partial_j (B_j \Phi)\big)(s)\,ds\;.$$

Since $\|\Phi(s)\|_{L^\infty} \lesssim M s^{-\kappa/2}$ by assumption and since the operator norm of $P_{t-s}\partial_j$ from L^∞ to itself is of order $(t-s)^{-1/2}$, it is straightforward from (7) and (9) to bound the integral appearing in this expression by (some multiple of) $M^c (t^{1-\frac{3\kappa}{2}} + t^{\frac{1}{2}-\kappa})$. The required bound $\|\Phi_0\|_{\mathcal{C}^{-\kappa}} \lesssim M^c$ follows at once, completing the proof. $\square$